SHORT GLOSSARY OF MATHEMATICS

Numbers
– natural, whole, integer
– rational, irrational
– fraction, numerator, denominator
– real
– imaginary
– complex
– positive, negative
– even, odd

Operations
– sum, to add, plus
– difference, to subtract, minus
– product, to multiply, times
– quotient, to divide, over
– power, to raise to the power *n*, to the *n*th, base, exponent
– inverse
– square, to square
– cube, to cube

Functions
– root, to take the *n*th root, radicand, index
– square root, to take the square root
– cube root, to take the cube root
– absolute value
– exponential, logarithm
– sine, cosine, tangent, cotangent
– domain, codomain
– argument
– injective (one-to-one), surjective (onto), bijective
– increasing, decreasing

Relations
– to equal, to be equal to
– to be (strictly) less than (or equal to)
– to be (strictly) greater than (or equal to)
– to be different from, to be distinct from
– to be equivalent to, to amount to

Expressions
– formula (*pl.* formulae)
– (to open/close) parenthesis (*pl.* parentheses), bracket
– to calculate, to compute
– to simplify
– to factor, to factor into
– to factor out, to pull out
– to rearrange, to group, to isolate
– to rewrite, to transform
– to cancel out
– to multiply straight across (numerator to numerator, denominator to denominator)
– to cross cancel
– to swap (the direction of the inequality symbol)
– to substitute
– to find
– to prove, to show, to verify
– to plot, to graph

Algebra
– variable
– value, constant
– equation, inequality
– left-hand side, right-hand side
– solution, to solve
– system (of equations, of inequalities)
– polynomial
– term, free term
– coefficient
– parameter
– degree
– linear, quadratic

Geometry
– Cartesian plane
– quadrant
– intersection
– symmetry
– opposite, adjacent

Measures
– distance
– length
– ratio
– perimeter, area

Geometric shapes
– point
– (straight) line
– curve
– conic section, conic
– locus (*pl.* loci)
– ray, half-line
– (line) segment
– angle
– polygon

Points
– coordinate
– *x*-coordinate, *y*-coordinate
– origin
– end point (of a line segment)
– midpoint

Lines
– axis (*pl.* axes)
– slope
– intercept
– horizontal, vertical
– parallel, perpendicular
– tangent, secant

Angles
– side
– vertex
– acute, right, obtuse, straight, reflex, full
– complementary, supplementary, explementary
– adjacent, vertical
– central, inscribed
– degrees, radians

Polygons
– triangle
– quadrilateral
– parallelogram
– trapezoid
– rectangle
– rhombus (*pl.* rhombi)
– square
– pentagon
– hexagon
– inscribed, circumscribed

Triangles
– side
– scalene, isosceles, equilateral
– right
– altitude, median, bisector
– leg, hypothenuse
– Pythagorean theorem

Rectangles
– base
– height
– diagonal

Conics
– circle, circumference
– ellipse
– parabola
– hyperbola
– focus (*pl.* foci)
– directrix (*pl.* directrices)
– eccentricity
– center
– vertex (*pl.* vertices)
– radius (*pl.* radii)
– diameter
– chord
– arc
– (semi-)minor axis, (semi-)major axis
– principal axis, transverse axis, conjugate axis
– branch
– asymptote

REVIEW OF CONIC SECTIONS

In this section we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. They are called **conic sections**, or **conics**, because they result from intersecting a cone with a plane as shown in Figure 1.

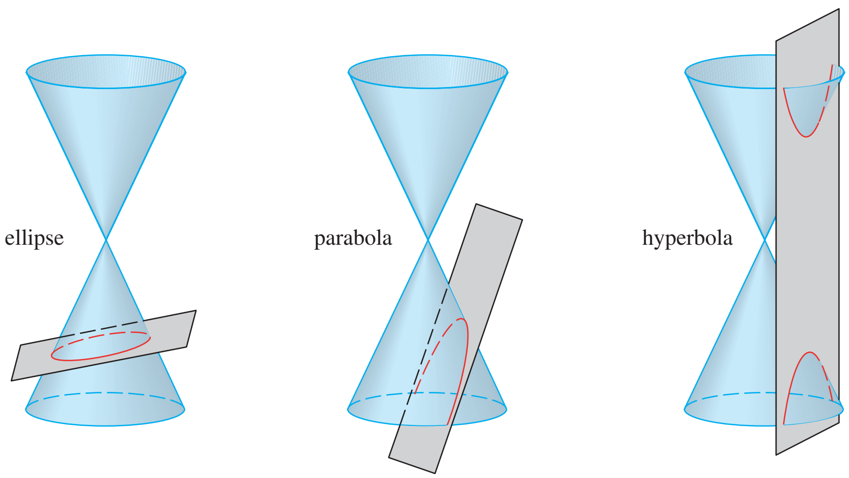


FIGURE 1
Conics

PARABOLAS

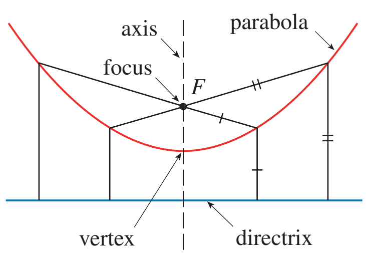


FIGURE 2

A **parabola** is the set of points in a plane that are equidistant from a fixed point F (called the **focus**) and a fixed line (called the **directrix**). This definition is illustrated by Figure 2. Notice that the point halfway between the focus and the directrix lies on the parabola; it is called the **vertex**. The line through the focus perpendicular to the directrix is called the **axis** of the parabola.

In the 16th century Galileo showed that the path of a projectile that is shot into the air at an angle to the ground is a parabola. Since then, parabolic shapes have been used in designing automobile headlights, reflecting telescopes, and suspension bridges. (See Challenge Problem 2.14 for the reflection property of parabolas that makes them so useful.)

We obtain a particularly simple equation for a parabola if we place its vertex at the origin O and its directrix parallel to the x-axis as in Figure 3. If the focus is the point $(0, p)$, then the directrix has the equation $y = -p$. If $P(x, y)$ is any point on the parabola, then the distance from P to the focus is

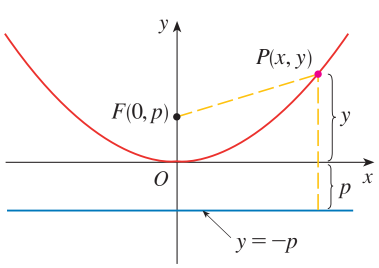


FIGURE 3

$$|PF| = \sqrt{x^2 + (y - p)^2}$$

and the distance from P to the directrix is $|y + p|$. (Figure 3 illustrates the case where $p > 0$.) The defining property of a parabola is that these distances are equal:

$$\sqrt{x^2 + (y - p)^2} = |y + p|$$

We get an equivalent equation by squaring and simplifying:

$$x^2 + (y - p)^2 = |y + p|^2 = (y + p)^2$$

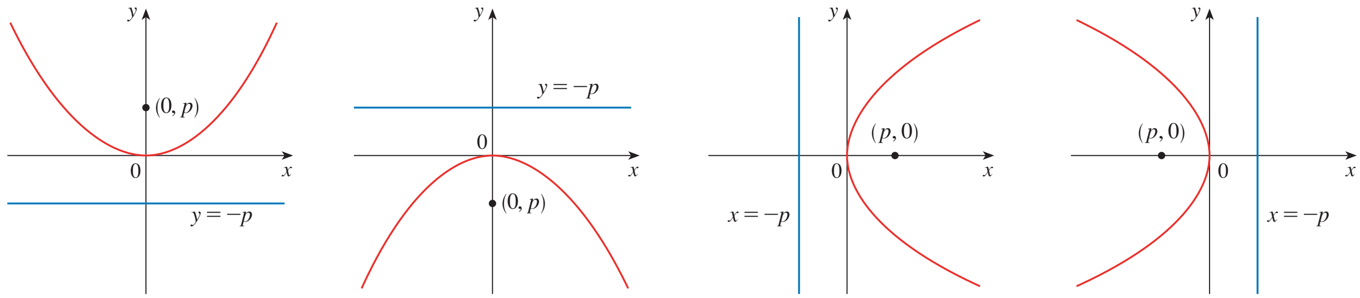
$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

$$x^2 = 4py$$

1 An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is

$$x^2 = 4py$$

If we write $a = 1/(4p)$, then the standard equation of a parabola (1) becomes $y = ax^2$. It opens upward if $p > 0$ and downward if $p < 0$ [see Figure 4, parts (a) and (b)]. The graph is symmetric with respect to the y-axis because (1) is unchanged when x is replaced by $-x$.



(a) $x^2 = 4py$, $p > 0$ (b) $x^2 = 4py$, $p < 0$ (c) $y^2 = 4px$, $p > 0$ (d) $y^2 = 4px$, $p < 0$

FIGURE 4

If we interchange x and y in (1), we obtain

$$\boxed{2} \qquad y^2 = 4px$$

which is an equation of the parabola with focus $(p, 0)$ and directrix $x = -p$. (Interchanging x and y amounts to reflecting about the diagonal line $y = x$.) The parabola opens to the right if $p > 0$ and to the left if $p < 0$ [see Figure 4, parts (c) and (d)]. In both cases the graph is symmetric with respect to the x-axis, which is the axis of the parabola.

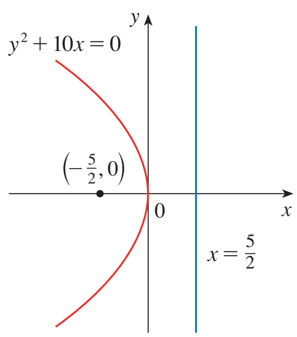


FIGURE 5

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 + 10x = 0$ and sketch the graph.

SOLUTION If we write the equation as $y^2 = -10x$ and compare it with Equation 2, we see that $4p = -10$, so $p = -\frac{5}{2}$. Thus the focus is $(p, 0) = \left(-\frac{5}{2}, 0\right)$ and the directrix is $x = \frac{5}{2}$. The sketch is shown in Figure 5. ■

ELLIPSES

An **ellipse** is the set of points in a plane the sum of whose distances from two fixed points F_1 and F_2 is a constant (see Figure 6). These two fixed points are called the **foci** (plural of **focus**). One of Kepler's laws is that the orbits of the planets in the solar system are ellipses with the Sun at one focus.

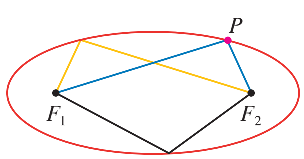


FIGURE 6

In order to obtain the simplest equation for an ellipse, we place the foci on the x-axis at the points $(-c, 0)$ and $(c, 0)$ as in Figure 7 so that the origin is halfway between the foci. Let the sum of the distances from a point on the ellipse to the foci be $2a > 0$. Then $P(x, y)$ is a point on the ellipse when

$$|PF_1| + |PF_2| = 2a$$

that is,
$$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a$$

or
$$\sqrt{(x - c)^2 + y^2} = 2a - \sqrt{(x + c)^2 + y^2}$$

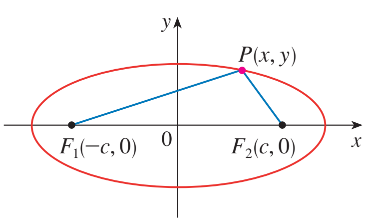


FIGURE 7

Squaring both sides, we have

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x + c)^2 + y^2} + x^2 + 2cx + c^2 + y^2$$

which simplifies to
$$a\sqrt{(x + c)^2 + y^2} = a^2 + cx$$

We square again:

$$a^2(x^2 + 2cx + c^2 + y^2) = a^4 + 2a^2cx + c^2x^2$$

which becomes
$$(a^2 - c^2)x^2 + a^2y^2 = a^2(a^2 - c^2)$$

From triangle F_1F_2P in Figure 7 we see that $2c < 2a$, so $c < a$ and, therefore, $a^2 - c^2 > 0$. For convenience, let $b^2 = a^2 - c^2$. Then the equation of the ellipse becomes $b^2x^2 + a^2y^2 = a^2b^2$ or, if both sides are divided by a^2b^2,

3 $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

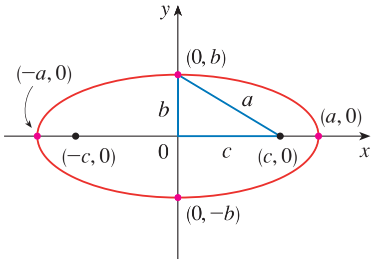


FIGURE 8

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Since $b^2 = a^2 - c^2 < a^2$, it follows that $b < a$. The x-intercepts are found by setting $y = 0$. Then $x^2/a^2 = 1$, or $x^2 = a^2$, so $x = \pm a$. The corresponding points $(a, 0)$ and $(-a, 0)$ are called the **vertices** of the ellipse and the line segment joining the vertices is called the **major axis**. To find the y-intercepts we set $x = 0$ and obtain $y^2 = b^2$, so $y = \pm b$. Equation 3 is unchanged if x is replaced by $-x$ or y is replaced by $-y$, so the ellipse is symmetric about both axes. Notice that if the foci coincide, then $c = 0$, so $a = b$ and the ellipse becomes a circle with radius $r = a = b$.

We summarize this discussion as follows (see also Figure 8).

4 The ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a \geqslant b > 0$$

has foci $(\pm c, 0)$, where $c^2 = a^2 - b^2$, and vertices $(\pm a, 0)$.

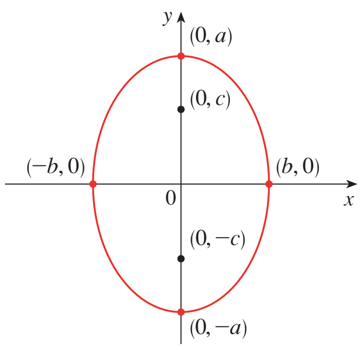


FIGURE 9

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1,\ a \geqslant b$

If the foci of an ellipse are located on the y-axis at $(0, \pm c)$, then we can find its equation by interchanging x and y in (4). (See Figure 9.)

5 The ellipse

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \qquad a \geqslant b > 0$$

has foci $(0, \pm c)$, where $c^2 = a^2 - b^2$, and vertices $(0, \pm a)$.

EXAMPLE 2 Sketch the graph of $9x^2 + 16y^2 = 144$ and locate the foci.

SOLUTION Divide both sides of the equation by 144:

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

The equation is now in the standard form for an ellipse, so we have $a^2 = 16$, $b^2 = 9$, $a = 4$, and $b = 3$. The x-intercepts are ± 4 and the y-intercepts are ± 3. Also, $c^2 = a^2 - b^2 = 7$, so $c = \sqrt{7}$ and the foci are $(\pm\sqrt{7}, 0)$. The graph is sketched in Figure 10. ■

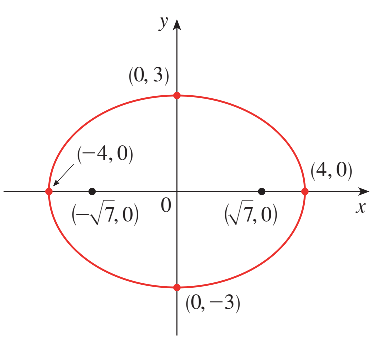


FIGURE 10

$9x^2 + 16y^2 = 144$

EXAMPLE 3 Find an equation of the ellipse with foci $(0, \pm 2)$ and vertices $(0, \pm 3)$.

SOLUTION Using the notation of (5), we have $c = 2$ and $a = 3$. Then we obtain $b^2 = a^2 - c^2 = 9 - 4 = 5$, so an equation of the ellipse is

$$\frac{x^2}{5} + \frac{y^2}{9} = 1$$

Another way of writing the equation is $9x^2 + 5y^2 = 45$.. ■

Like parabolas, ellipses have an interesting reflection property that has practical consequences. If a source of light or sound is placed at one focus of a surface with elliptical cross-sections, then all the light or sound is reflected off the surface to the other focus (see

Exercise 59). This principle is used in *lithotripsy*, a treatment for kidney stones. A reflector with elliptical cross-section is placed in such a way that the kidney stone is at one focus. High-intensity sound waves generated at the other focus are reflected to the stone and destroy it without damaging surrounding tissue. The patient is spared the trauma of surgery and recovers within a few days.

HYPERBOLAS

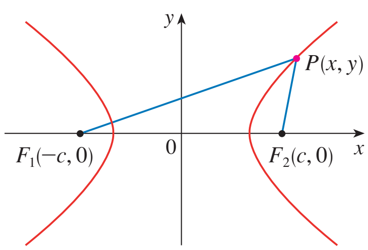


FIGURE 11
P is on the hyperbola when $|PF_1| - |PF_2| = \pm 2a$

A **hyperbola** is the set of all points in a plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is a constant. This definition is illustrated in Figure 11.

Hyperbolas occur frequently as graphs of equations in chemistry, physics, biology, and economics (Boyle's Law, Ohm's Law, supply and demand curves). A particularly significant application of hyperbolas is found in the navigation systems developed in World Wars I and II (see Exercise 51).

Notice that the definition of a hyperbola is similar to that of an ellipse; the only change is that the sum of distances has become a difference of distances. In fact, the derivation of the equation of a hyperbola is also similar to the one given earlier for an ellipse. It is left as Exercise 52 to show that when the foci are on the x-axis at $(\pm c, 0)$ and the difference of distances is $|PF_1| - |PF_2| = \pm 2a$, then the equation of the hyperbola is

6 $$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where $c^2 = a^2 + b^2$. Notice that the x-intercepts are again $\pm a$ and the points $(a, 0)$ and $(-a, 0)$ are the **vertices** of the hyperbola. But if we put $x = 0$ in Equation 6 we get $y^2 = -b^2$, which is impossible, so there is no y-intercept. The hyperbola is symmetric with respect to both axes.

To analyze the hyperbola further, we look at Equation 6 and obtain

$$\frac{x^2}{a^2} = 1 + \frac{y^2}{b^2} \geqslant 1$$

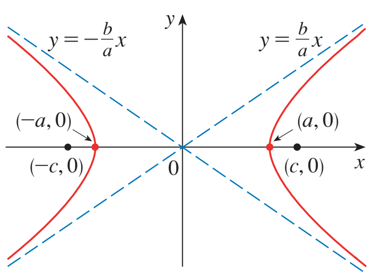


FIGURE 12
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

This shows that $x^2 \geqslant a^2$, so $|x| = \sqrt{x^2} \geqslant a$. Therefore, we have $x \geqslant a$ or $x \leqslant -a$. This means that the hyperbola consists of two parts, called its *branches*.

When we draw a hyperbola it is useful to first draw its **asymptotes**, which are the dashed lines $y = (b/a)x$ and $y = -(b/a)x$ shown in Figure 12. Both branches of the hyperbola approach the asymptotes; that is, they come arbitrarily close to the asymptotes.

7 The hyperbola
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
has foci $(\pm c, 0)$, where $c^2 = a^2 + b^2$, vertices $(\pm a, 0)$, and asymptotes $y = \pm(b/a)x$.

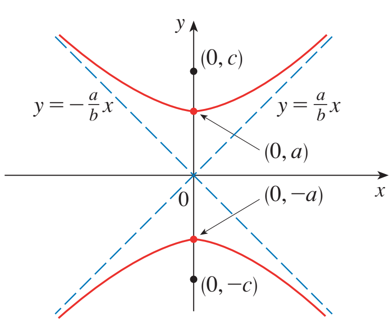


FIGURE 13
$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$

If the foci of a hyperbola are on the y-axis, then by reversing the roles of x and y we obtain the following information, which is illustrated in Figure 13.

8 The hyperbola
$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$
has foci $(0, \pm c)$, where $c^2 = a^2 + b^2$, vertices $(0, \pm a)$, and asymptotes $y = \pm(a/b)x$.

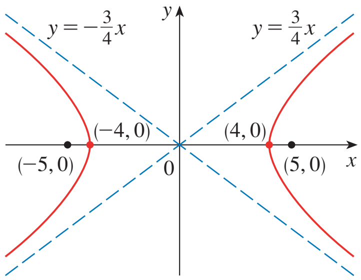


FIGURE 14
$9x^2 - 16y^2 = 144$

EXAMPLE 4 Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

SOLUTION If we divide both sides of the equation by 144, it becomes

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

which is of the form given in (7) with $a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$, the foci are $(\pm 5, 0)$. The asymptotes are the lines $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$. The graph is shown in Figure 14. ■

EXAMPLE 5 Find the foci and equation of the hyperbola with vertices $(0, \pm 1)$ and asymptote $y = 2x$.

SOLUTION From (8) and the given information, we see that $a = 1$ and $a/b = 2$. Thus, $b = a/2 = \frac{1}{2}$ and $c^2 = a^2 + b^2 = \frac{5}{4}$. The foci are $\left(0, \pm\sqrt{5}/2\right)$ and the equation of the hyperbola is

$$y^2 - 4x^2 = 1$$ ■

SHIFTED CONICS

We shift conics by taking the standard equations (1), (2), (4), (5), (7), and (8) and replacing x and y by $x - h$ and $y - k$.

EXAMPLE 6 Find an equation of the ellipse with foci $(2, -2)$, $(4, -2)$ and vertices $(1, -2)$, $(5, -2)$.

SOLUTION The major axis is the line segment that joins the vertices $(1, -2)$, $(5, -2)$ and has length 4, so $a = 2$. The distance between the foci is 2, so $c = 1$. Thus, $b^2 = a^2 - c^2 = 3$. Since the center of the ellipse is $(3, -2)$, we replace x and y in (4) by $x - 3$ and $y + 2$ to obtain

$$\frac{(x-3)^2}{4} + \frac{(y+2)^2}{3} = 1$$

as the equation of the ellipse. ■

EXAMPLE 7 Sketch the conic

$$9x^2 - 4y^2 - 72x + 8y + 176 = 0$$

and find its foci.

SOLUTION We complete the squares as follows:

$$4(y^2 - 2y) - 9(x^2 - 8x) = 176$$
$$4(y^2 - 2y + 1) - 9(x^2 - 8x + 16) = 176 + 4 - 144$$
$$4(y-1)^2 - 9(x-4)^2 = 36$$
$$\frac{(y-1)^2}{9} - \frac{(x-4)^2}{4} = 1$$

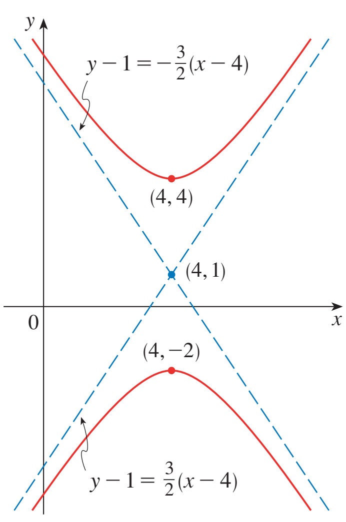


FIGURE 15
$9x^2 - 4y^2 - 72x + 8y + 176 = 0$

This is in the form (8) except that x and y are replaced by $x - 4$ and $y - 1$. Thus, $a^2 = 9$, $b^2 = 4$, and $c^2 = 13$. The hyperbola is shifted four units to the right and one unit upward. The foci are $\left(4, 1 + \sqrt{13}\right)$ and $\left(4, 1 - \sqrt{13}\right)$ and the vertices are $(4, 4)$ and $(4, -2)$. The asymptotes are $y - 1 = \pm\frac{3}{2}(x - 4)$. The hyperbola is sketched in Figure 15. ■

EXERCISES

A Click here for answers. S Click here for solutions.

1–8 ■ Find the vertex, focus, and directrix of the parabola and sketch its graph.

1. $x = 2y^2$

2. $4y + x^2 = 0$

3. $4x^2 = -y$

4. $y^2 = 12x$

5. $(x + 2)^2 = 8(y - 3)$

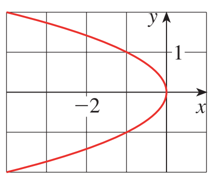
6. $x - 1 = (y + 5)^2$

7. $y^2 + 2y + 12x + 25 = 0$

8. $y + 12x - 2x^2 = 16$

9–10 ■ Find an equation of the parabola. Then find the focus and directrix.

9.



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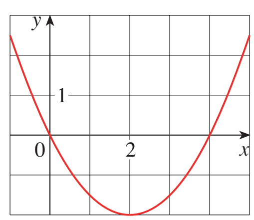


11–16 ■ Find the vertices and foci of the ellipse and sketch its graph.

11. $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$

12. $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1$

13. $4x^2 + y^2 = 16$

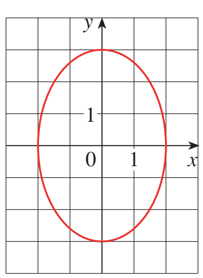
14. $4x^2 + 25y^2 = 25$

15. $9x^2 - 18x + 4y^2 = 27$

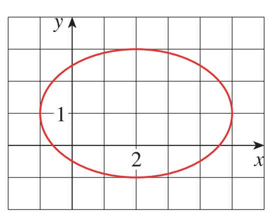
16. $x^2 + 2y^2 - 6x + 4y + 7 = 0$

17–18 ■ Find an equation of the ellipse. Then find its foci.

17.



18.



19–20 ■ Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1$

20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1$

21. $y^2 - x^2 = 4$

22. $9x^2 - 4y^2 = 36$

23. $2y^2 - 3x^2 - 4y + 12x + 8 = 0$

24. $16x^2 - 9y^2 + 64x - 90y = 305$

25–30 ■ Identify the type of conic section whose equation is given and find the vertices and foci.

25. $x^2 = y + 1$

26. $x^2 = y^2 + 1$

27. $x^2 = 4y - 2y^2$

28. $y^2 - 8y = 6x - 16$

29. $y^2 + 2y = 4x^2 + 3$

30. $4x^2 + 4x + y^2 = 0$

31–48 ■ Find an equation for the conic that satisfies the given conditions.

31. Parabola, vertex $(0, 0)$, focus $(0, -2)$

32. Parabola, vertex $(1, 0)$, directrix $x = -5$

33. Parabola, focus $(-4, 0)$, directrix $x = 2$

34. Parabola, focus $(3, 6)$, vertex $(3, 2)$

35. Parabola, vertex $(0, 0)$, axis the x-axis, passing through $(1, -4)$

36. Parabola, vertical axis, passing through $(-2, 3)$, $(0, 3)$, and $(1, 9)$

37. Ellipse, foci $(\pm 2, 0)$, vertices $(\pm 5, 0)$

38. Ellipse, foci $(0, \pm 5)$, vertices $(0, \pm 13)$

39. Ellipse, foci $(0, 2)$, $(0, 6)$ vertices $(0, 0)$, $(0, 8)$

40. Ellipse, foci $(0, -1)$, $(8, -1)$, vertex $(9, -1)$

41. Ellipse, center $(2, 2)$, focus $(0, 2)$, vertex $(5, 2)$

42. Ellipse, foci $(\pm 2, 0)$, passing through $(2, 1)$

43. Hyperbola, foci $(0, \pm 3)$, vertices $(0, \pm 1)$

44. Hyperbola, foci $(\pm 6, 0)$, vertices $(\pm 4, 0)$

45. Hyperbola, foci $(1, 3)$ and $(7, 3)$, vertices $(2, 3)$ and $(6, 3)$

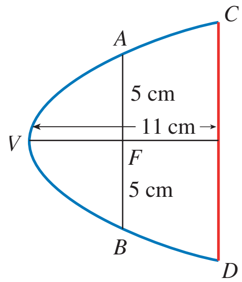
46. Hyperbola, foci $(2, -2)$ and $(2, 8)$, vertices $(2, 0)$ and $(2, 6)$

47. Hyperbola, vertices $(\pm 3, 0)$, asymptotes $y = \pm 2x$

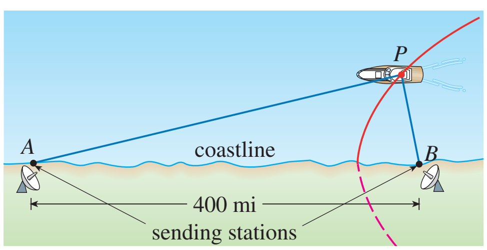
48. Hyperbola, foci $(2, 2)$ and $(6, 2)$, asymptotes $y = x - 2$ and $y = 6 - x$

49. The point in a lunar orbit nearest the surface of the moon is called *perilune* and the point farthest from the surface is called *apolune*. The *Apollo 11* spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

50. A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.
(a) Find an equation of the parabola.
(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.



51. In the LORAN (LOng RAnge Navigation) radio navigation system, two radio stations located at A and B transmit simultaneous signals to a ship or an aircraft located at P. The onboard computer converts the time difference in receiving these signals into a distance difference $|PA| - |PB|$, and this, according to the definition of a hyperbola, locates the ship or aircraft on one branch of a hyperbola (see the figure). Suppose that station B is located 400 mi due east of station A on a coastline. A ship received the signal from B 1200 microseconds (μs) before it received the signal from A.
(a) Assuming that radio signals travel at a speed of 980 ft/μs, find an equation of the hyperbola on which the ship lies.
(b) If the ship is due north of B, how far off the coastline is the ship?



52. Use the definition of a hyperbola to derive Equation 6 for a hyperbola with foci $(\pm c, 0)$ and vertices $(\pm a, 0)$.

53. Show that the function defined by the upper branch of the hyperbola $y^2/a^2 - x^2/b^2 = 1$ is concave upward.

54. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

55. Determine the type of curve represented by the equation

$$\frac{x^2}{k} + \frac{y^2}{k - 16} = 1$$

in each of the following cases: (a) $k > 16$, (b) $0 < k < 16$, and (c) $k < 0$.
(d) Show that all the curves in parts (a) and (b) have the same foci, no matter what the value of k is.

56. (a) Show that the equation of the tangent line to the parabola $y^2 = 4px$ at the point (x_0, y_0) can be written as $y_0 y = 2p(x + x_0)$.
(b) What is the x-intercept of this tangent line? Use this fact to draw the tangent line.

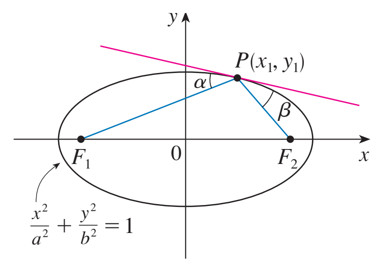
57. Use Simpson's Rule with $n = 10$ to estimate the length of the ellipse $x^2 + 4y^2 = 4$.

58. The planet Pluto travels in an elliptical orbit around the Sun (at one focus). The length of the major axis is 1.18×10^{10} km and the length of the minor axis is 1.14×10^{10} km. Use Simpson's Rule with $n = 10$ to estimate the distance traveled by the planet during one complete orbit around the Sun.

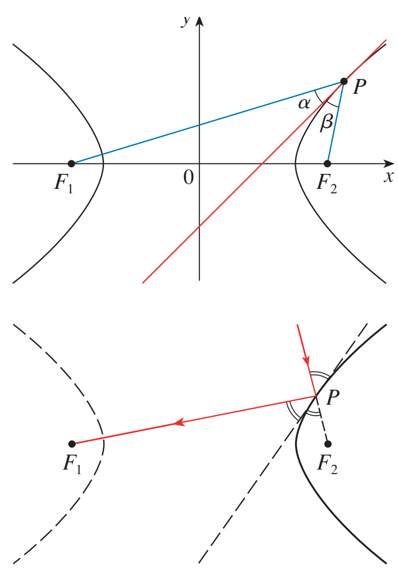
59. Let $P(x_1, y_1)$ be a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the ellipse as in the figure. Prove that $\alpha = \beta$. This explains how whispering galleries and lithotripsy work. Sound coming from one focus is reflected and passes through the other focus. [*Hint:* Use the formula

$$\tan \alpha = \frac{m_2 - m_1}{1 + m_2 m_1}$$

to show that $\tan \alpha = \tan \beta$. See **Challenge Problem 2.13**.]



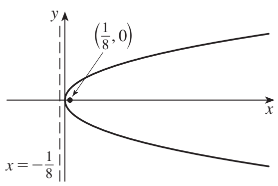
60. Let $P(x_1, y_1)$ be a point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$ with foci F_1 and F_2 and let α and β be the angles between the lines PF_1, PF_2 and the hyperbola as shown in the figure. Prove that $\alpha = \beta$. (This is the reflection property of the hyperbola. It shows that light aimed at a focus F_2 of a hyperbolic mirror is reflected toward the other focus F_1.)



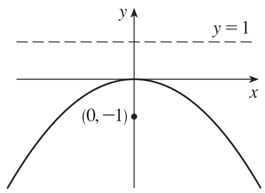
ANSWERS

S Click here for solutions.

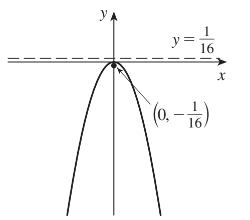
1. $(0, 0)$, $\left(\frac{1}{8}, 0\right)$, $x = -\frac{1}{8}$



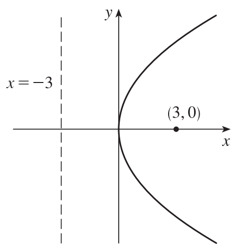
2. $(0, 0)$, $(0, -1)$, $y = 1$



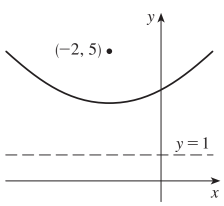
3. $(0, 0)$, $\left(0, -\frac{1}{16}\right)$, $y = \frac{1}{16}$



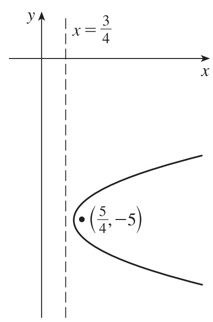
4. $(0, 0)$, $(3, 0)$, $x = -3$



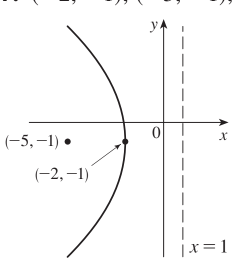
5. $(-2, 3)$, $(-2, 5)$, $y = 1$



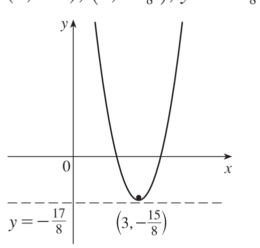
6. $(1, -5)$, $\left(\frac{5}{4}, -5\right)$, $x = \frac{3}{4}$



7. $(-2, -1)$, $(-5, -1)$, $x = 1$



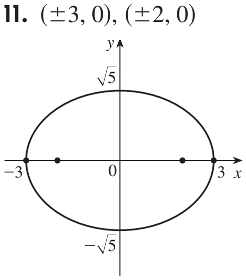
8. $(3, -2)$, $\left(3, -\frac{15}{8}\right)$, $y = -\frac{17}{8}$



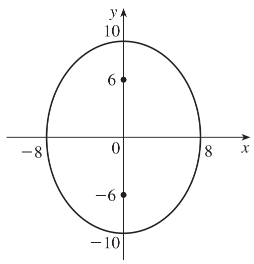
9. $x = -y^2$, focus $\left(-\frac{1}{4}, 0\right)$, directrix $x = \frac{1}{4}$

10. $(x - 2)^2 = 2(y + 2)$, focus $\left(2, -\frac{3}{2}\right)$, directrix $y = -\frac{5}{2}$

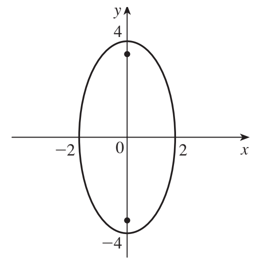
11. $(\pm 3, 0)$, $(\pm 2, 0)$



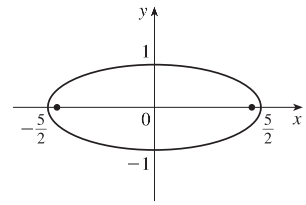
12. $(0, \pm 10)$, $(0, \pm 6)$



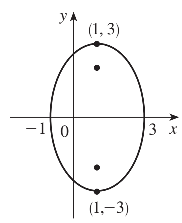
13. $(0, \pm 4)$, $\left(0, \pm 2\sqrt{3}\right)$



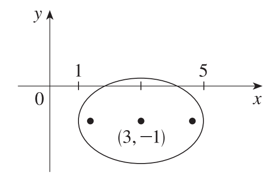
14. $\left(\pm\frac{5}{2}, 0\right)$, $\left(\pm\frac{\sqrt{21}}{2}, 0\right)$



15. $(1, \pm 3)$, $\left(1, \pm\sqrt{5}\right)$



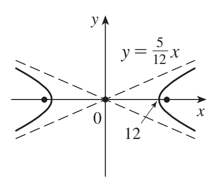
16. $(1, -1)$ and $(5, -1)$, $\left(3 \pm \sqrt{2}, -1\right)$



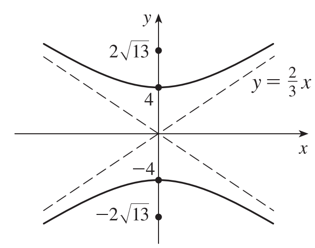
17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$, foci $\left(0, \pm\sqrt{5}\right)$

18. $\dfrac{(x - 2)^2}{9} + \dfrac{(y - 1)^2}{4} = 1$, foci $\left(2 \pm \sqrt{5}, 1\right)$

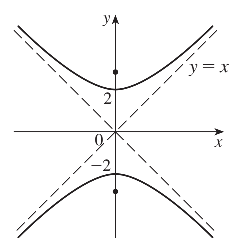
19. $(\pm 12, 0)$, $(\pm 13, 0)$,
$y = \pm\frac{5}{12}x$



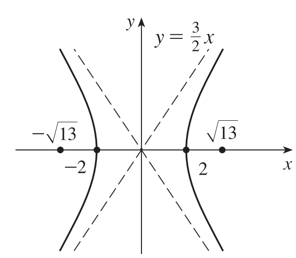
20. $(0, \pm 4)$, $\left(0, \pm 2\sqrt{13}\right)$,
$y = \pm\frac{2}{3}x$



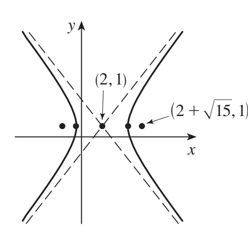
21. $(0, \pm 2)$, $\left(0, \pm 2\sqrt{2}\right)$,
$y = \pm x$



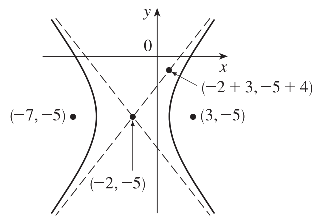
22. $(\pm 2, 0)$, $\left(\pm\sqrt{13}, 0\right)$,
$y = \pm\frac{3}{2}x$



23. $\left(2 \pm \sqrt{6}, 1\right)$, $\left(2 \pm \sqrt{15}, 1\right)$,
$y - 1 = \pm\left(\sqrt{6}/2\right)(x - 2)$



24. $(-5, -5)$ and $(1, -5)$, $(-7, -5)$ and $(3, -5)$, $y + 5 = \pm\frac{4}{3}(x + 2)$



25. Parabola, $(0, -1)$, $\left(0, -\frac{3}{4}\right)$ **26.** Hyperbola, $(\pm 1, 0)$, $\left(\pm\sqrt{2}, 0\right)$

27. Ellipse, $\left(\pm\sqrt{2}, 1\right)$, $(\pm 1, 1)$ **28.** Parabola, $(0, 4)$, $\left(\frac{3}{2}, 4\right)$

29. Hyperbola, $(0, 1)$, $(0, -3)$; $\left(0, -1 \pm \sqrt{5}\right)$

30. Ellipse, $\left(-\frac{1}{2}, \pm 1\right)$, $\left(-\frac{1}{2}, \pm\sqrt{3}/2\right)$

31. $x^2 = -8y$ **32.** $y^2 = 24(x - 1)$ **33.** $y^2 = -12(x + 1)$

34. $(x - 3)^2 = 16(y - 2)$ **35.** $y^2 = 16x$

36. $2x^2 + 4x - y + 3 = 0$ **37.** $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$

38. $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$ **39.** $\dfrac{x^2}{12} + \dfrac{(y - 4)^2}{16} = 1$

40. $\dfrac{(x - 4)^2}{25} + \dfrac{(y + 1)^2}{9} = 1$ **41.** $\dfrac{(x - 2)^2}{9} + \dfrac{(y - 2)^2}{5} = 1$

42. $\dfrac{2x^2}{9 + \sqrt{17}} + \dfrac{2y^2}{1 + \sqrt{17}} = 1$

43. $y^2 - \frac{1}{8}x^2 = 1$ **44.** $\frac{1}{16}x^2 - \frac{1}{20}y^2 = 1$

45. $\dfrac{(x - 4)^2}{4} - \dfrac{(y - 3)^2}{5} = 1$

46. $\frac{1}{9}(y - 3)^2 - \frac{1}{16}(x - 2)^2 = 1$ **47.** $\frac{1}{9}x^2 - \frac{1}{36}y^2 = 1$

48. $\frac{1}{2}(x - 4)^2 - \frac{1}{2}(y - 2)^2 = 1$

49. $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$

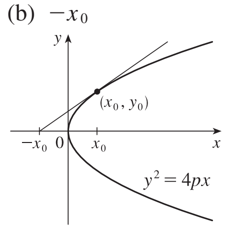
50. (a) $p = \frac{5}{2}$, $y^2 = 10x$ (b) $2\sqrt{110}$

51. (a) $\dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$ (b) $\approx$248 mi

54. $3x^2 - 2xy + 3y^2 = 8$

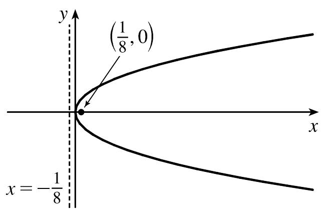
55. (a) Ellipse (b) Hyperbola (c) No curve

56. (b) $-x_0$ **57.** 9.69 **58.** 3.64×10^{10} km

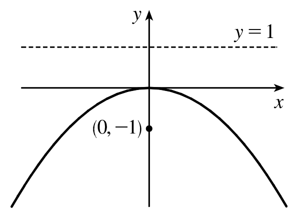


SOLUTIONS

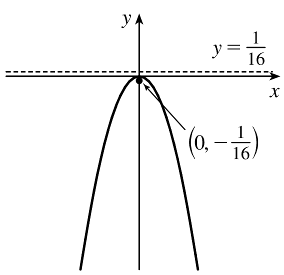
1. $x = 2y^2 \quad \Rightarrow \quad y^2 = \frac{1}{2}x$. $4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(0,0)$, the focus is $\left(\frac{1}{8}, 0\right)$, and the directrix is $x = -\frac{1}{8}$.



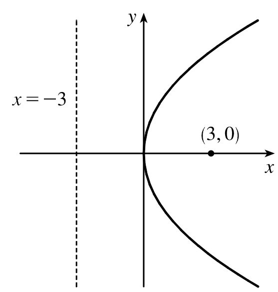
2. $4y + x^2 = 0 \quad \Rightarrow \quad x^2 = -4y$. $4p = -4$, so $p = -1$. The vertex is $(0,0)$, the focus is $(0,-1)$, and the directrix is $y = 1$.



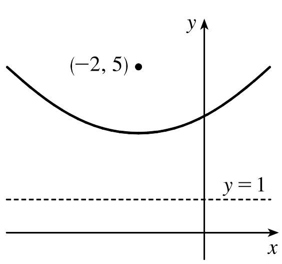
3. $4x^2 = -y \quad \Rightarrow \quad x^2 = -\frac{1}{4}y$. $4p = -\frac{1}{4}$, so $p = -\frac{1}{16}$. The vertex is $(0,0)$, the focus is $\left(0, -\frac{1}{16}\right)$, and the directrix is $y = \frac{1}{16}$.



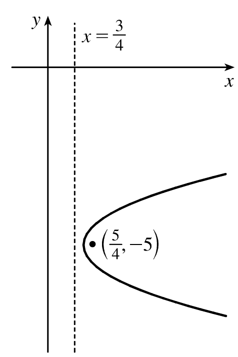
4. $y^2 = 12x$. $4p = 12$, so $p = 3$. The vertex is $(0,0)$, the focus is $(3,0)$, and the directrix is $x = -3$.



5. $(x+2)^2 = 8(y-3)$. $4p = 8$, so $p = 2$. The vertex is $(-2,3)$, the focus is $(-2,5)$, and the directrix is $y = 1$.



6. $x - 1 = (y+5)^2$. $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(1,-5)$, the focus is $\left(\frac{5}{4}, -5\right)$, and the directrix is $x = \frac{3}{4}$.

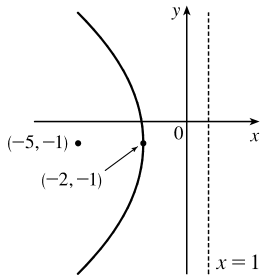


7. $y^2 + 2y + 12x + 25 = 0 \quad \Rightarrow$

$y^2 + 2y + 1 = -12x - 24 \quad \Rightarrow$

$(y+1)^2 = -12(x+2)$. $4p = -12$, so $p = -3$.

The vertex is $(-2, -1)$, the focus is $(-5, -1)$, and the directrix is $x = 1$.



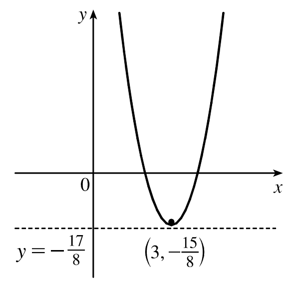
8. $y + 12x - 2x^2 = 16 \quad \Rightarrow$

$2x^2 - 12x = y - 16 \quad \Rightarrow$

$2(x^2 - 6x + 9) = y - 16 + 18 \quad \Rightarrow$

$2(x-3)^2 = y + 2 \quad \Rightarrow \quad (x-3)^2 = \frac{1}{2}(y+2)$.

$4p = \frac{1}{2}$, so $p = \frac{1}{8}$. The vertex is $(3, -2)$, the focus is $\left(3, -\frac{15}{8}\right)$, and the directrix is $y = -\frac{17}{8}$.

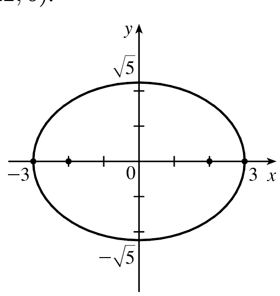


9. The equation has the form $y^2 = 4px$, where $p < 0$. Since the parabola passes through $(-1, 1)$, we have $1^2 = 4p(-1)$, so $4p = -1$ and an equation is $y^2 = -x$ or $x = -y^2$. $4p = -1$, so $p = -\frac{1}{4}$ and the focus is $\left(-\frac{1}{4}, 0\right)$ while the directrix is $x = \frac{1}{4}$.

10. The vertex is $(2, -2)$, so the equation is of the form $(x-2)^2 = 4p(y+2)$, where $p > 0$. The point $(0, 0)$ is on the parabola, so $4 = 4p(2)$ and $4p = 2$. Thus, an equation is $(x-2)^2 = 2(y+2)$. $4p = 2$, so $p = \frac{1}{2}$ and the focus is $\left(2, -\frac{3}{2}\right)$ while the directrix is $y = -\frac{5}{2}$.

11. $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1 \quad \Rightarrow \quad a = \sqrt{9} = 3, b = \sqrt{5}$,

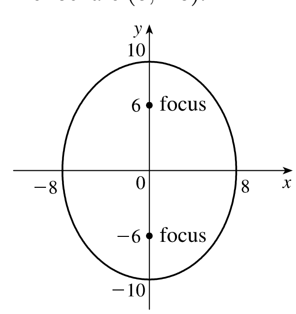
$c = \sqrt{a^2 - b^2} = \sqrt{9-5} = 2$. The ellipse is centered at $(0, 0)$, with vertices at $(\pm 3, 0)$. The foci are $(\pm 2, 0)$.



12. $\dfrac{x^2}{64} + \dfrac{y^2}{100} = 1 \quad \Rightarrow \quad a = \sqrt{100} = 10$,

$b = \sqrt{64} = 8$, $c = \sqrt{a^2 - b^2} = \sqrt{100 - 64} = 6$.

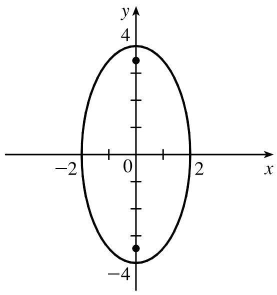
The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 10)$. The foci are $(0, \pm 6)$.



13. $4x^2 + y^2 = 16 \quad \Rightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1 \quad \Rightarrow$

$a = \sqrt{16} = 4$, $b = \sqrt{4} = 2$,

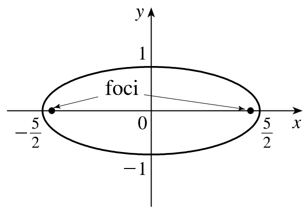
$c = \sqrt{a^2 - b^2} = \sqrt{16 - 4} = 2\sqrt{3}$. The ellipse is centered at $(0, 0)$, with vertices at $(0, \pm 4)$. The foci are $\left(0, \pm 2\sqrt{3}\right)$.



14. $4x^2 + 25y^2 = 25 \quad \Rightarrow \quad \dfrac{x^2}{25/4} + \dfrac{y^2}{1} = 1 \quad \Rightarrow$

$a = \sqrt{\frac{25}{4}} = \frac{5}{2}$, $b = \sqrt{1} = 1$,

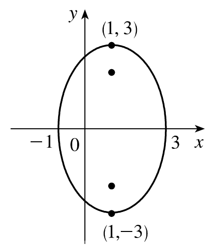
$c = \sqrt{a^2 - b^2} = \sqrt{\frac{25}{4} - 1} = \sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$. The ellipse is centered at $(0, 0)$, with vertices at $\left(\pm\frac{5}{2}, 0\right)$. The foci are $\left(\pm\frac{\sqrt{21}}{2}, 0\right)$.



15. $9x^2 - 18x + 4y^2 = 27 \quad \Leftrightarrow \quad 9(x^2 - 2x + 1) + 4y^2 = 27 + 9 \quad \Leftrightarrow$

$9(x - 1)^2 + 4y^2 = 36 \quad \Leftrightarrow \quad \dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1 \quad \Rightarrow \quad a = 3, b = 2,$

$c = \sqrt{5} \quad \Rightarrow \quad$ center $(1, 0)$, vertices $(1, \pm 3)$, foci $\left(1, \pm\sqrt{5}\right)$

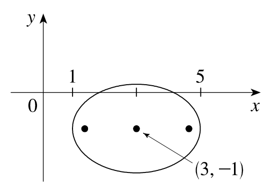


16. $x^2 - 6x + 2y^2 + 4y = -7 \quad \Leftrightarrow$

$x^2 - 6x + 9 + 2(y^2 + 2y + 1) = -7 + 9 + 2 \quad \Leftrightarrow$

$(x - 3)^2 + 2(y + 1)^2 = 4 \quad \Leftrightarrow$

$\dfrac{(x-3)^2}{4} + \dfrac{(y+1)^2}{2} = 1 \quad \Rightarrow \quad a = 2, b = \sqrt{2} = c \quad \Rightarrow \quad$ center $(3, -1)$, vertices $(1, -1)$ and $(5, -1)$, foci $\left(3 \pm \sqrt{2}, -1\right)$



17. The center is $(0, 0)$, $a = 3$, and $b = 2$, so an equation is $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$. $c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(0, \pm\sqrt{5}\right)$.

18. The ellipse is centered at $(2, 1)$, with $a = 3$ and $b = 2$. An equation is $\dfrac{(x-2)^2}{9} + \dfrac{(y-1)^2}{4} = 1$.

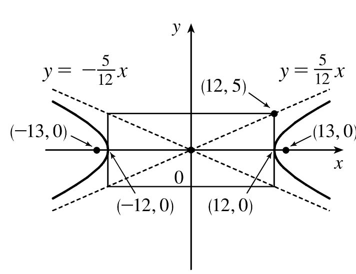
$c = \sqrt{a^2 - b^2} = \sqrt{5}$, so the foci are $\left(2 \pm \sqrt{5}, 1\right)$.

19. $\dfrac{x^2}{144} - \dfrac{y^2}{25} = 1 \quad \Rightarrow \quad a = 12, b = 5, c = \sqrt{144 + 25} = 13 \quad \Rightarrow$

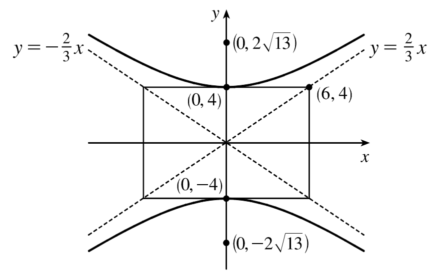
center $(0, 0)$, vertices $(\pm 12, 0)$, foci $(\pm 13, 0)$,

asymptotes $y = \pm\frac{5}{12}x$.

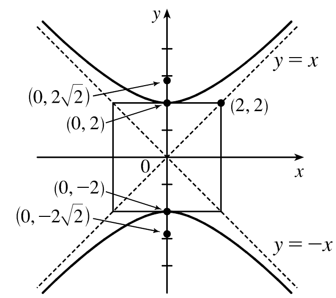
Note: It is helpful to draw a $2a$-by-$2b$ rectangle whose center is the center of the hyperbola. The asymptotes are the extended diagonals of the rectangle.



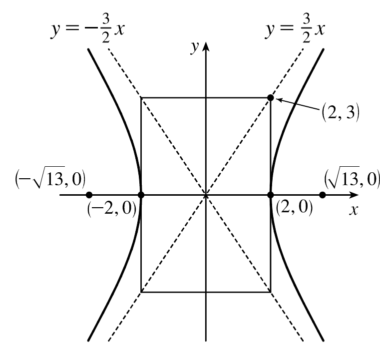
20. $\dfrac{y^2}{16} - \dfrac{x^2}{36} = 1 \quad \Rightarrow \quad a = 4, b = 6,$
$c = \sqrt{a^2 + b^2} = \sqrt{16 + 36} = \sqrt{52} = 2\sqrt{13}$. The center is $(0, 0)$, the vertices are $(0, \pm 4)$, the foci are $\left(0, \pm 2\sqrt{13}\right)$, and the asymptotes are the lines $y = \pm \frac{a}{b} x = \pm \frac{2}{3} x$.



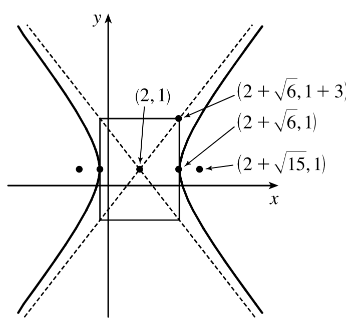
21. $y^2 - x^2 = 4 \quad \Leftrightarrow \quad \dfrac{y^2}{4} - \dfrac{x^2}{4} = 1 \quad \Rightarrow \quad a = \sqrt{4} = 2 = b,$
$c = \sqrt{4 + 4} = 2\sqrt{2} \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(0, \pm 2)$, foci $\left(0, \pm 2\sqrt{2}\right)$, asymptotes $y = \pm x$



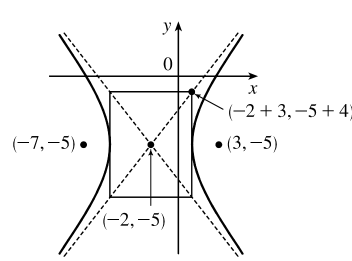
22. $9x^2 - 4y^2 = 36 \quad \Leftrightarrow \quad \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1 \quad \Rightarrow \quad a = \sqrt{4} = 2,$
$b = \sqrt{9} = 3, c = \sqrt{4 + 9} = \sqrt{13} \quad \Rightarrow \quad$ center $(0, 0)$, vertices $(\pm 2, 0)$, foci $\left(\pm\sqrt{13}, 0\right)$, asymptotes $y = \pm \frac{3}{2} x$



23. $2y^2 - 4y - 3x^2 + 12x = -8 \quad \Leftrightarrow$
$2(y^2 - 2y + 1) - 3(x^2 - 4x + 4) = -8 + 2 - 12 \quad \Leftrightarrow$
$2(y - 1)^2 - 3(x - 2)^2 = -18 \quad \Leftrightarrow \quad \dfrac{(x - 2)^2}{6} - \dfrac{(y - 1)^2}{9} = 1$
$\Rightarrow \quad a = \sqrt{6}, b = 3, c = \sqrt{15} \quad \Rightarrow \quad$ center $(2, 1)$, vertices $\left(2 \pm \sqrt{6}, 1\right)$, foci $\left(2 \pm \sqrt{15}, 1\right)$, asymptotes $y - 1 = \pm \frac{3}{\sqrt{6}} (x - 2)$ or $y - 1 = \pm \dfrac{\sqrt{6}}{2} (x - 2)$



24. $16x^2 + 64x - 9y^2 - 90y = 305 \quad \Leftrightarrow$
$16(x^2 + 4x + 4) - 9(y^2 + 10y + 25) = 305 + 64 - 225 \quad \Leftrightarrow$
$16(x + 2)^2 - 9(y + 5)^2 = 144 \quad \Leftrightarrow \quad \dfrac{(x + 2)^2}{9} - \dfrac{(y + 5)^2}{16} = 1$
$\Rightarrow \quad a = 3, b = 4, c = 5 \quad \Rightarrow \quad$ center $(-2, -5)$, vertices $(-5, -5)$ and $(1, -5)$, foci $(-7, -5)$ and $(3, -5)$, asymptotes $y + 5 = \pm \frac{4}{3} (x + 2)$



25. $x^2 = y + 1 \quad \Leftrightarrow \quad x^2 = 1(y + 1)$. This is an equation of a *parabola* with $4p = 1$, so $p = \frac{1}{4}$. The vertex is $(0, -1)$ and the focus is $\left(0, -\frac{3}{4}\right)$.

26. $x^2 = y^2 + 1 \quad \Leftrightarrow \quad x^2 - y^2 = 1$. This is an equation of a *hyperbola* with vertices $(\pm 1, 0)$. The foci are at $\left(\pm\sqrt{1+1}, 0\right) = \left(\pm\sqrt{2}, 0\right)$.

27. $x^2 = 4y - 2y^2 \quad \Leftrightarrow \quad x^2 + 2y^2 - 4y = 0 \quad \Leftrightarrow \quad x^2 + 2(y^2 - 2y + 1) = 2 \quad \Leftrightarrow \quad x^2 + 2(y-1)^2 = 2 \quad \Leftrightarrow$ $\dfrac{x^2}{2} + \dfrac{(y-1)^2}{1} = 1$. This is an equation of an *ellipse* with vertices at $\left(\pm\sqrt{2}, 1\right)$. The foci are at $\left(\pm\sqrt{2-1}, 1\right) = (\pm 1, 1)$.

28. $y^2 - 8y = 6x - 16 \quad \Leftrightarrow \quad y^2 - 8y + 16 = 6x \quad \Leftrightarrow \quad (y-4)^2 = 6x$. This is an equation of a *parabola* with $4p = 6$, so $p = \frac{3}{2}$. The vertex is $(0, 4)$ and the focus is $\left(\frac{3}{2}, 4\right)$.

29. $y^2 + 2y = 4x^2 + 3 \quad \Leftrightarrow \quad y^2 + 2y + 1 = 4x^2 + 4 \quad \Leftrightarrow \quad (y+1)^2 - 4x^2 = 4 \quad \Leftrightarrow \quad \dfrac{(y+1)^2}{4} - x^2 = 1$. This is an equation of a *hyperbola* with vertices $(0, -1 \pm 2) = (0, 1)$ and $(0, -3)$. The foci are at $\left(0, -1 \pm \sqrt{4+1}\right) = \left(0, -1 \pm \sqrt{5}\right)$.

30. $4x^2 + 4x + y^2 = 0 \quad \Leftrightarrow \quad 4\left(x^2 + x + \frac{1}{4}\right) + y^2 = 1 \quad \Leftrightarrow \quad 4\left(x + \frac{1}{2}\right)^2 + y^2 = 1 \quad \Leftrightarrow \quad \dfrac{\left(x + \frac{1}{2}\right)^2}{1/4} + y^2 = 1$. This is an equation of an *ellipse* with vertices $\left(-\frac{1}{2}, 0 \pm 1\right) = \left(-\frac{1}{2}, \pm 1\right)$. The foci are at $\left(-\frac{1}{2}, 0 \pm \sqrt{1 - \frac{1}{4}}\right) = \left(-\frac{1}{2}, \pm\sqrt{3}/2\right)$.

31. The parabola with vertex $(0, 0)$ and focus $(0, -2)$ opens downward and has $p = -2$, so its equation is $x^2 = 4py = -8y$.

32. The parabola with vertex $(1, 0)$ and directrix $x = -5$ opens to the right and has $p = 6$, so its equation is $y^2 = 4p(x-1) = 24(x-1)$.

33. The distance from the focus $(-4, 0)$ to the directrix $x = 2$ is $2 - (-4) = 6$, so the distance from the focus to the vertex is $\frac{1}{2}(6) = 3$ and the vertex is $(-1, 0)$. Since the focus is to the left of the vertex, $p = -3$. An equation is $y^2 = 4p(x+1) \quad \Rightarrow \quad y^2 = -12(x+1)$.

34. The distance from the focus $(3, 6)$ to the vertex $(3, 2)$ is $6 - 2 = 4$. Since the focus is above the vertex, $p = 4$. An equation is $(x-3)^2 = 4p(y-2) \quad \Rightarrow \quad (x-3)^2 = 16(y-2)$.

35. The parabola must have equation $y^2 = 4px$, so $(-4)^2 = 4p(1) \quad \Rightarrow \quad p = 4 \quad \Rightarrow \quad y^2 = 16x$.

36. Vertical axis $\quad \Rightarrow \quad (x-h)^2 = 4p(y-k)$. Substituting $(-2, 3)$ and $(0, 3)$ gives $(-2-h)^2 = 4p(3-k)$ and $(-h)^2 = 4p(3-k) \quad \Rightarrow \quad (-2-h)^2 = (-h)^2 \quad \Rightarrow \quad 4 + 4h + h^2 = h^2 \quad \Rightarrow \quad h = -1 \quad \Rightarrow \quad 1 = 4p(3-k)$. Substituting $(1, 9)$ gives $[1 - (-1)]^2 = 4p(9-k) \quad \Rightarrow \quad 4 = 4p(9-k)$. Solving for p from these equations gives $p = \dfrac{1}{4(3-k)} = \dfrac{1}{9-k} \quad \Rightarrow \quad 4(3-k) = 9-k \quad \Rightarrow \quad k = 1 \quad \Rightarrow \quad p = \frac{1}{8} \quad \Rightarrow \quad (x+1)^2 = \frac{1}{2}(y-1) \quad \Rightarrow$ $2x^2 + 4x - y + 3 = 0$.

37. The ellipse with foci $(\pm 2, 0)$ and vertices $(\pm 5, 0)$ has center $(0, 0)$ and a horizontal major axis, with $a = 5$ and $c = 2$, so $b = \sqrt{a^2 - c^2} = \sqrt{21}$. An equation is $\dfrac{x^2}{25} + \dfrac{y^2}{21} = 1$.

38. The ellipse with foci $(0, \pm 5)$ and vertices $(0, \pm 13)$ has center $(0, 0)$ and a vertical major axis, with $c = 5$ and $a = 13$, so $b = \sqrt{a^2 - c^2} = 12$. An equation is $\dfrac{x^2}{144} + \dfrac{y^2}{169} = 1$.

39. Since the vertices are $(0, 0)$ and $(0, 8)$, the ellipse has center $(0, 4)$ with a vertical axis and $a = 4$. The foci at $(0, 2)$ and $(0, 6)$ are 2 units from the center, so $c = 2$ and $b = \sqrt{a^2 - c^2} = \sqrt{4^2 - 2^2} = \sqrt{12}$. An equation is $\dfrac{(x-0)^2}{b^2} + \dfrac{(y-4)^2}{a^2} = 1 \quad \Rightarrow \quad \dfrac{x^2}{12} + \dfrac{(y-4)^2}{16} = 1$.

40. Since the foci are $(0, -1)$ and $(8, -1)$, the ellipse has center $(4, -1)$ with a horizontal axis and $c = 4$. The vertex $(9, -1)$ is 5 units from the center, so $a = 5$ and $b = \sqrt{a^2 - c^2} = \sqrt{5^2 - 4^2} = \sqrt{9}$. An equation is $\dfrac{(x-4)^2}{a^2} + \dfrac{(y+1)^2}{b^2} = 1 \quad \Rightarrow \quad \dfrac{(x-4)^2}{25} + \dfrac{(y+1)^2}{9} = 1$.

41. Center $(2, 2)$, $c = 2$, $a = 3 \quad \Rightarrow \quad b = \sqrt{5} \quad \Rightarrow \quad \frac{1}{9}(x-2)^2 + \frac{1}{5}(y-2)^2 = 1$

42. Center $(0, 0)$, $c = 2$, major axis horizontal $\quad \Rightarrow \quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and $b^2 = a^2 - c^2 = a^2 - 4$. Since the ellipse passes through $(2, 1)$, we have $2a = |PF_1| + |PF_2| = \sqrt{17} + 1 \quad \Rightarrow \quad a^2 = \frac{9 + \sqrt{17}}{2}$ and $b^2 = \frac{1 + \sqrt{17}}{2}$, so the ellipse has equation $\dfrac{2x^2}{9 + \sqrt{17}} + \dfrac{2y^2}{1 + \sqrt{17}} = 1$.

43. Center $(0, 0)$, vertical axis, $c = 3$, $a = 1 \quad \Rightarrow \quad b = \sqrt{8} = 2\sqrt{2} \quad \Rightarrow \quad y^2 - \frac{1}{8}x^2 = 1$

44. Center $(0, 0)$, horizontal axis, $c = 6$, $a = 4 \quad \Rightarrow \quad b = 2\sqrt{5} \quad \Rightarrow \quad \frac{1}{16}x^2 - \frac{1}{20}y^2 = 1$

45. Center $(4, 3)$, horizontal axis, $c = 3$, $a = 2 \quad \Rightarrow \quad b = \sqrt{5} \quad \Rightarrow \quad \frac{1}{4}(x-4)^2 - \frac{1}{5}(y-3)^2 = 1$

46. Center $(2, 3)$, vertical axis, $c = 5$, $a = 3 \quad \Rightarrow \quad b = 4 \quad \Rightarrow \quad \frac{1}{9}(y-3)^2 - \frac{1}{16}(x-2)^2 = 1$

47. Center $(0, 0)$, horizontal axis, $a = 3$, $\frac{b}{a} = 2 \quad \Rightarrow \quad b = 6 \quad \Rightarrow \quad \frac{1}{9}x^2 - \frac{1}{36}y^2 = 1$

48. Center $(4, 2)$, horizontal axis, asymptotes $y - 2 = \pm(x - 4) \quad \Rightarrow \quad c = 2$, $b/a = 1 \quad \Rightarrow \quad a = b \quad \Rightarrow$ $c^2 = 4 = a^2 + b^2 = 2a^2 \quad \Rightarrow \quad a^2 = 2 \quad \Rightarrow \quad \frac{1}{2}(x-4)^2 - \frac{1}{2}(y-2)^2 = 1$

49. In Figure 8, we see that the point on the ellipse closest to a focus is the closer vertex (which is a distance $a - c$ from it) while the farthest point is the other vertex (at a distance of $a + c$). So for this lunar orbit, $(a - c) + (a + c) = 2a = (1728 + 110) + (1728 + 314)$, or $a = 1940$; and $(a + c) - (a - c) = 2c = 314 - 110$, or $c = 102$. Thus, $b^2 = a^2 - c^2 = 3{,}753{,}196$, and the equation is $\dfrac{x^2}{3{,}763{,}600} + \dfrac{y^2}{3{,}753{,}196} = 1$.

50. (a) Choose V to be the origin, with x-axis through V and F. Then F is $(p, 0)$, A is $(p, 5)$, so substituting A into the equation $y^2 = 4px$ gives $25 = 4p^2$ so $p = \frac{5}{2}$ and $y^2 = 10x$.

(b) $x = 11 \quad \Rightarrow \quad y = \sqrt{110} \quad \Rightarrow \quad |CD| = 2\sqrt{110}$

51. (a) Set up the coordinate system so that A is $(-200, 0)$ and B is $(200, 0)$.
$|PA| - |PB| = (1200)(980) = 1{,}176{,}000$ ft $= \frac{2450}{11}$ mi $= 2a \quad \Rightarrow \quad a = \frac{1225}{11}$, and $c = 200$ so $b^2 = c^2 - a^2 = \frac{3{,}339{,}375}{121} \quad \Rightarrow \quad \dfrac{121x^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1$.

(b) Due north of $B \quad \Rightarrow \quad x = 200 \quad \Rightarrow \quad \dfrac{(121)(200)^2}{1{,}500{,}625} - \dfrac{121y^2}{3{,}339{,}375} = 1 \quad \Rightarrow \quad y = \dfrac{133{,}575}{539} \approx 248$ mi

52. $|PF_1| - |PF_2| = \pm 2a \quad \Leftrightarrow \quad \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a \quad \Leftrightarrow$

$\sqrt{(x+c)^2 + y^2} = \sqrt{(x-c)^2 + y^2} \pm 2a \quad \Leftrightarrow \quad (x+c)^2 + y^2 = (x-c)^2 + y^2 + 4a^2 \pm 4a\sqrt{(x-c)^2 + y^2}$

$\Leftrightarrow \quad 4cx - 4a^2 = \pm 4a\sqrt{(x-c)^2 + y^2} \quad \Leftrightarrow \quad c^2x^2 - 2a^2cx + a^4 = a^2\left(x^2 - 2cx + c^2 + y^2\right) \quad \Leftrightarrow$

$\left(c^2 - a^2\right)x^2 - a^2y^2 = a^2\left(c^2 - a^2\right) \quad \Leftrightarrow \quad b^2x^2 - a^2y^2 = a^2b^2$ (where $b^2 = c^2 - a^2$) $\quad \Leftrightarrow \quad \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

53. The function whose graph is the upper branch of this hyperbola is concave upward. The function is $y = f(x) = a\sqrt{1 + \dfrac{x^2}{b^2}} = \dfrac{a}{b}\sqrt{b^2 + x^2}$, so $y' = \dfrac{a}{b}x\left(b^2 + x^2\right)^{-1/2}$ and

$y'' = \dfrac{a}{b}\left[\left(b^2 + x^2\right)^{-1/2} - x^2\left(b^2 + x^2\right)^{-3/2}\right] = ab\left(b^2 + x^2\right)^{-3/2} > 0$ for all x, and so f is concave upward.

54. We can follow exactly the same sequence of steps as in the derivation of Formula 4, except we use the points $(1, 1)$ and $(-1, -1)$ in the distance formula (first equation of that derivation) so $\sqrt{(x-1)^2 + (y-1)^2} + \sqrt{(x+1)^2 + (y+1)^2} = 4$ will lead (after moving the second term to the right, squaring, and simplifying) to $2\sqrt{(x+1)^2 + (y+1)^2} = x + y + 4$, which, after squaring and simplifying again, leads to $3x^2 - 2xy + 3y^2 = 8$.

55. (a) If $k > 16$, then $k - 16 > 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is an *ellipse* since it is the sum of two squares on the left side.

(b) If $0 < k < 16$, then $k - 16 < 0$, and $\dfrac{x^2}{k} + \dfrac{y^2}{k-16} = 1$ is a *hyperbola* since it is the difference of two squares on the left side.

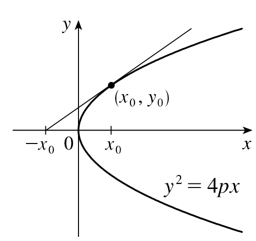
(c) If $k < 0$, then $k - 16 < 0$, and there is *no curve* since the left side is the sum of two negative terms, which cannot equal 1.

(d) In case (a), $a^2 = k$, $b^2 = k - 16$, and $c^2 = a^2 - b^2 = 16$, so the foci are at $(\pm 4, 0)$. In case (b), $k - 16 < 0$, so $a^2 = k$, $b^2 = 16 - k$, and $c^2 = a^2 + b^2 = 16$, and so again the foci are at $(\pm 4, 0)$.

56. (a) $y^2 = 4px \quad \Rightarrow \quad 2yy' = 4p \quad \Rightarrow \quad y' = \dfrac{2p}{y}$, so the tangent line is

$y - y_0 = \dfrac{2p}{y_0}(x - x_0) \quad \Rightarrow \quad yy_0 - y_0^2 = 2p(x - x_0) \quad \Leftrightarrow$

$yy_0 - 4px_0 = 2px - 2px_0 \quad \Rightarrow \quad yy_0 = 2p(x + x_0)$.



(b) The x-intercept is $-x_0$.

57. Use the parametrization $x = 2\cos t$, $y = \sin t$, $0 \le t \le 2\pi$ to get

$$L = 4\int_0^{\pi/2}\sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt = 4\int_0^{\pi/2}\sqrt{4\sin^2 t + \cos^2 t}\,dt = 4\int_0^{\pi/2}\sqrt{3\sin^2 t + 1}\,dt$$

Using Simpson's Rule with $n = 10$, $\Delta t = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(t) = \sqrt{3\sin^2 t + 1}$, we get

$$L \approx \tfrac{4}{3}\left(\tfrac{\pi}{20}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 2f\left(\tfrac{8\pi}{20}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 9.69$$

58. The length of the major axis is $2a$, so $a = \frac{1}{2}(1.18 \times 10^{10}) = 5.9 \times 10^9$. The length of the minor axis is $2b$, so $b = \frac{1}{2}(1.14 \times 10^{10}) = 5.7 \times 10^9$. An equation of the ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, or converting into parametric equations, $x = a\cos\theta$ and $y = b\sin\theta$. So

$$L = 4\int_0^{\pi/2} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2}\, d\theta = 4\int_0^{\pi/2} \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\, d\theta$$

Using Simpson's Rule with $n = 10$, $\Delta\theta = \frac{\pi/2 - 0}{10} = \frac{\pi}{20}$, and $f(\theta) = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$, we get

$$\begin{aligned} L &\approx 4 \cdot S_{10} \\ &= 4 \cdot \tfrac{\pi}{20 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 2f\left(\tfrac{8\pi}{20}\right) + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \\ &\approx 3.64 \times 10^{10} \text{ km} \end{aligned}$$

59. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \;\Rightarrow\; \dfrac{2x}{a^2} + \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = -\dfrac{b^2x}{a^2y}$ $(y \neq 0)$. Thus, the slope of the tangent line at P is $-\dfrac{b^2x_1}{a^2y_1}$. The slope of F_1P is $\dfrac{y_1}{x_1 + c}$ and of F_2P is $\dfrac{y_1}{x_1 - c}$. By the formula from Problems Plus, we have

$$\tan\alpha = \frac{\dfrac{y_1}{x_1+c} + \dfrac{b^2x_1}{a^2y_1}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{a^2y_1^2 + b^2x_1(x_1+c)}{a^2y_1(x_1+c) - b^2x_1y_1} = \frac{a^2b^2 + b^2cx_1}{c^2x_1y_1 + a^2cy_1} \quad \left[\begin{array}{c}\text{using } b^2x_1^2 + a^2y_1^2 = a^2b^2 \\ \text{and } a^2 - b^2 = c^2\end{array}\right]$$

$$= \frac{b^2(cx_1 + a^2)}{cy_1(cx_1 + a^2)} = \frac{b^2}{cy_1}$$

and

$$\tan\beta = \frac{-\dfrac{y_1}{x_1-c} - \dfrac{b^2x_1}{a^2y_1}}{1 - \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-a^2y_1^2 - b^2x_1(x_1-c)}{a^2y_1(x_1-c) - b^2x_1y_1} = \frac{-a^2b^2 + b^2cx_1}{c^2x_1y_1 - a^2cy_1} = \frac{b^2(cx_1 - a^2)}{cy_1(cx_1 - a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

60. The slopes of the line segments F_1P and F_2P are $\dfrac{y_1}{x_1 + c}$ and $\dfrac{y_1}{x_1 - c}$, where P is (x_1, y_1). Differentiating implicitly, $\dfrac{2x}{a^2} - \dfrac{2yy'}{b^2} = 0 \;\Rightarrow\; y' = \dfrac{b^2x}{a^2y} \;\Rightarrow\;$ the slope of the tangent at P is $\dfrac{b^2x_1}{a^2y_1}$, so by the formula from Problems Plus,

$$\tan\alpha = \frac{\dfrac{b^2x_1}{a^2y_1} - \dfrac{y_1}{x_1+c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1+c)}} = \frac{b^2x_1(x_1+c) - a^2y_1^2}{a^2y_1(x_1+c) + b^2x_1y_1}$$

$$= \frac{b^2(cx_1 + a^2)}{cy_1(cx_1 + a^2)} \quad \left[\begin{array}{c}\text{using } x_1^2/a^2 - y_1^2/b^2 = 1 \\ \text{and } a^2 + b^2 = c^2\end{array}\right] \quad = \frac{b^2}{cy_1}$$

and

$$\tan\beta = \frac{-\dfrac{b^2x_1}{a^2y_1} + \dfrac{y_1}{x_1-c}}{1 + \dfrac{b^2x_1y_1}{a^2y_1(x_1-c)}} = \frac{-b^2x_1(x_1-c) + a^2y_1^2}{a^2y_1(x_1-c) + b^2x_1y_1} = \frac{b^2(cx_1 - a^2)}{cy_1(cx_1 - a^2)} = \frac{b^2}{cy_1}$$

So $\alpha = \beta$.

How LORAN works

https://timeandnavigation.si.edu/multimedia-asset/how-loran-works

This is a shortened version of the 1947 "LORAN for Ocean Navigation" filmstrip produced by the Coast Guard as a sales pitch to commercial shipping lines to adopt LORAN (as a both a navigational aid and to assist in distress situations). Updated with a new narration track in place of the distorted period track, the film provides a brief overview of the operational theory behind LORAN.

Transcript:

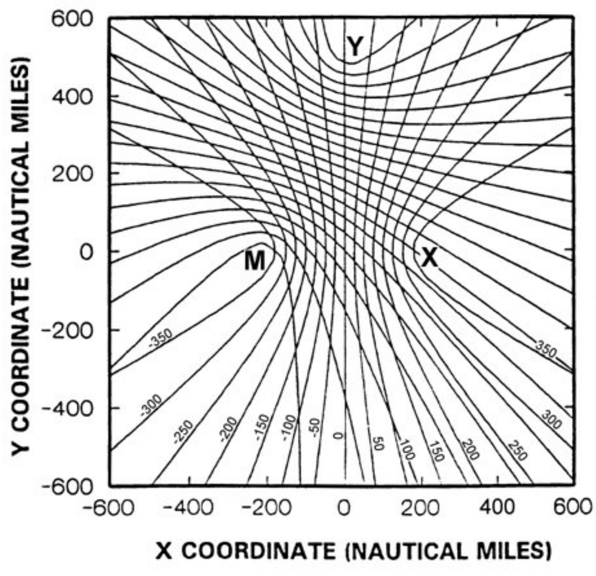
Operating 24 hours a day, numerous nations maintain a network of LORAN transmitting stations to service the major shipping lanes.

There are a pair here, for example, Siasconset and Bodie Island. One is known as the master station, the other is the slave. For a moment, consider these two stations operating as a synchronous pair, simultaneously emitting short pulses of radio energy. Leaving both shore stations simultaneously, a pair of pulses travel out into space, in all directions, at a constant speed, roughly 186,000 miles per second or the speed of light. Plus, the pulse from the closer station will reach the ship an instant before the pulse from the other station.

The LORAN ship-board gear measures this difference in time of arrival in millionths of a second, or microseconds. It simply determines how much longer one pulse takes to reach the ship than the other pulse.

Now, this same time interval will be observed at many points within the range of the two shore stations. And when connected these points form a hyperbola known as the LORAN line of position. To aid the navigator in obtaining a fix, specially prepared LORAN tables and charts contain accurately plotted lines of position on the various time differences encountered in a particular area.

Having one line of position we then obtain readings from another pair of stations. An accurate fix is established at the intersection of the two lines of position.



A family of hyperbolic lines generated by LORAN signals.

Conic sections – Theory

1. Fill in the gaps in the following sentences by choosing the most appropriate words from the box below. There are 7 extra words that you do not need to use.

a) A conic is a curve obtained as the __intersection__ between a cone and a __plane__. Every conic can also be defined as a particular geometric __locus__.

b) There are three types of conic. If the curve is closed, it is an ellipse. If it consists of two separate parts, called __branches__, it is a __hyperbola__. Otherwise it is a __parabola__.

c) The __circle__ is a special kind of __ellipse__, in which the two foci coincide with the __center__, and the axes have the same __length__, equal to the __diameter__.

d) A parabola can also be seen as the __graph__ of a function $y = ax^2 + bx + c$, with $a \neq 0$. Its axis of __symmetry__ is the line passing through the focus and __perpendicular__ to the directrix. The axis intersects the parabola in a point, called the __vertex__.

e) For any point on an ellipse, the __sum__ between the distances from the two foci is constant, and is also equal to the __major__ axis.

f) The eccentricity of a hyperbola is the number defined as the __ratio__ between the focal __distance__ and the __transverse__ axis. It is always __greater__ than one.

asymptotes	~~branches~~	~~center~~	~~circle~~	~~diameter~~	difference	~~distance~~
~~ellipse~~	focus	~~graph~~	~~greater~~	~~hyperbola~~	~~intersection~~	~~length~~
less	~~locus~~	~~major~~	minor	~~parabola~~	parallel	~~perpendicular~~
~~plane~~	radius	~~ratio~~	~~sum~~	~~symmetry~~	~~transverse~~	~~vertex~~

Conic sections – Exercises

2. Consider the ellipse with foci $F_1(0,-4)$ and $F_2(0,4)$, and major axis of length 10.

a) Determine the canonical equation of the ellipse.

b) Sketch the graph.

c) Find the eccentricity.

3. Consider the hyperbola of equation $\dfrac{x^2}{4} - y^2 = 1$.

a) Find the coordinates of the foci.

b) Determine the equations of the asymptotes.

c) Let r be the line of equation $x - 2y - 1 = 0$. Determine the position of r relative to the hyperbola. If there are any intersection points, find their coordinates.